Mean field methods for stochastic population dynamics

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Project summary:

Population dynamics are widely studied by using both deterministic and stochastic models that aim to describe interactions between large heterogeneous groups of individuals. Some of the objectives of these studies are: establishing the long term behaviour of the whole population (e.g., which species will thrive, and which ones will disappear), or determining the expected survival time of a particular species. Other interesting questions in community ecology that this PhD project can help answer include identifying which models predict a sensible relative species abundance (RSA), which is used to calculate the probability that a certain species has a given number of individuals alive. In studies of natural environments, it is important to answer questions of this kind as accurately as possible, in order to make informed decisions concerning human interventions aimed, e.g., at preserving fauna.

A substantial chunk of current research around population dynamics hinges on numerical methods and theoretical studies of various mathematical models. This is challenging because of the complexity of modelling interactions between large groups of individuals. In this project we will study emerging area of modelling in which individuals are not only affected by the rest of population but also subject to random perturbations of their state. The use of stochastic models is valuable since it is virtually impossible to include all possible factors that can affect the dynamic of a population in a tractable model. In general, the term "stochastic" is broadly understood as some sort of "randomness" in the system under consideration, see, for example, [2] and [6]. Recently developed methods of McKean-Vlasov equations as limits of the population dynamics when their size grows to infinity enables tractable studies of stochastic population models similarly as the master equation is used in classical models.

The candidate will study interacting particle systems that describe stochastic interaction of individuals and their mean-field limit from a mathematical perspective, hence using tools such as stochastic analysis, function spaces, probability, numerical analysis and so on. One should note that these models can describe a wide range of interacting individuals, from animals (as we see in more detail in the example below) to humans, from more complex human structures (such as banks) to neurons in the brain, or even inanimate particles (like gas).

Detailed Project Description:

Setting out the problem. We will need to consider very high dimensional systems of SDEs in order to describe the evolution of different individuals/groups of individuals/species (depending on the particular problem at hand). For simplicity, here we can think that each SDE in the system represents the evolution of one individual. One of the key challenges is of course that each individual is affected by its own current state as well as by the state of all other (or a sub-group of) individuals in the same environment. Let's consider the example of a group of foxes in a specific geographical region and say that they prey on hares from the same region. We could attach a stochastic equation to each fox in order to describe for example their level of strength/health and geographical position. The ability of a fox to keep in good health will depend on its ability to hunt hares and feed regularly. Clearly there is competition with the other foxes and it might as well be that a relatively weak fox, surrounded by stronger specimens, will perish. On the other hand, if all foxes are equally healthy at outset, then their geographical position will play an important role: foxes hunting in densely populated areas will face more competition than other ones hunting in more peripheral areas. Of course, this is a simplification and other factors must be included in our model as, e.g, density of preys etc. However, what immediately emerges is the complexity of describing interactions if the number of individuals is large.

Mean-field models. In this project we will use mathematical tools that have been attracting increasing attention over the past 10 years: namely we will concentrate on the so-called mean-field stochastic differential equations and related McKean-Vlasov models, see e.g [4, 5]. The main idea underpinning these equations is that a system with a large number of interacting individuals can be approximated by a macroscopic equation, provided that one can describe some sort of "average" interaction within the population (the so-called "mean-field approximation"). To be more precise, one needs to determine how each individual interacts on average with its peers/competitors. The task is mathematically non-trivial because in practice we need to solve SDEs which take into account the average of the population in a way which is consistent with the individual's behaviour. Moreover, one needs to be careful and prove rigorously that the mean-field approximation is indeed a good approximation of the initial population (which instead is finite). Therefore, it is crucial to determine, amongst other things, the rate of convergence of the approximation and how different types of mean-field models are suited to describe specific population dynamics. It is worth mentioning in passing that McKean-Vlasov equations can be interpreted as generalised mean-field equations where the interactions between many individuals is accounted for by considering other features than just an "average" interaction. For some details on the mathematical tools used see for example [1] and the recent monography [3].

Objectives and novelty:

The importance of this study stems primarily from two main considerations: (i) systems with a large number of interacting individuals are not tractable analytically; (ii) often they are prohibitively expensive to solve numerically. Hence the development of rigorous, model-consistent and tractable approximations is necessary both from the theoretical and numerical point of view.

This is a very new area of mathematics with many fundamental results still in need of development. It is important to emphasise that the study of mean-field equations (as well as McKean-Vlasov equations) has strong links with other areas of stochastic analysis, including backward SDEs, stochastic partial differential equations and new notions of differentiation of measure-valued functions. On the practical side, the development of numerical methods for mean-field equations is still limited and offers many potential directions of research, driven by applications in population dynamics.

The project will focus on the mathematical theory and tools underpinning the population model, in particular the student will

1. Formulate (in a rigorous mathematical way) a stochastic interacting particle system for a large population, that describes a class biological/social problems in the realm of population and behavioural ecology
2. Study this large interacting system, in particular find conditions under which the system has a solution which is unique. Study also the regularity properties of this solution
3. Postulate the mean-field (or McKean-Vlasov) approximating equation and solve it, that is study existence, uniqueness and regularity of the solution
4. Investigate the limiting behaviour of the large interacting system as the number of individuals tends to infinity with the aim to show that the limiting system is indeed the one postulated above. Study also the rate of convergence of the approximation
5. Devise numerical schemes to solve the system and the mean-field equation numerically, and show convergence of the numerical schemes Implement the numerical schemes

Learning Outcomes:

A student enrolling in this project will learn about several different aspects of mathematical modelling of biological dynamics, in particular stochastic population dynamics. These may include stochastic differential equation, numerical methods, stochastic interacting particle systems, mean-field games, stochastic control and stochastic partial differential equations.

After the completion of the PhD project, the student could then apply these powerful tools to stochastic modelling of population dynamics, or other areas of natural sciences (such as stochastic modelling for neuroscience or physics), or social dynamics of humans, or different areas such as financial mathematics.

Keywords:

Mathematics, modelling, stochastic population dynamics, mean-field equations, interacting particle systems, stochastic analysis.

Student Profile:

A candidate choosing this project should satisfy the following pre-requisites:

1. A degree in mathematics
2. Good knowledge of the basic results/methods from probability and statistics
3. Basic knowledge of stochastic differential equations
4. Interest in mathematical modelling of problems arising from applied areas such as biology, ecology, and population dynamics

References:

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